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POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR NTH ORDER Q-DIffERENTIAL EQUATIONS

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ABSTRACT

In this paper, we investigate the problem of existence of positive solutions for the nonlinear q-boundary value problem or quantum boundary value problem:

$$D_q^n u(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1,$$

satisfying three kinds of q-different boundary value conditions. Our analysis relies on Krasnoselskii's fixed point theorem of cone.

Keywords: *Q-difference equations; Fixed point theorem; Boundary value problem; Positive solution*

1.INTRODUCTION:

There is currently a great deal of interest in positive solutions for several types of boundary value problems. A large part of the literature on positive solutions to boundary value problems seems to be traced back to Krasnoselskii's work on nonlinear operator equations [15], especially the part dealing with the theory of cones in Banach spaces. In 1994, Erbe and Wang [6] applied Krasnoselskii's work to eigenvalue problems to establish intervals of the parameter λ for which there is at least one positive solution. In 1995, Eloe and Henderson [2] obtained the solutions that are positive to a cone for the boundary value problem

$$u^{(n)}(t) + a(t) f(u) = 0, \quad 0 < t < 1,$$
$$u^{(i)}(0) = u^{(n-2)}(1) = 0, 0 \le i \le n-2.$$

Since this pioneering works, a lot research has been done in this area [3, 6, 11, 16, 19, 20]. In 2008,EL-Shahed [4] obtained the existence of positive solutions to nonlinear nth order boundary value problems

$$u^{(n)}(t) + \lambda a(t) f(u(t)) = 0, 0 < t < 1,$$
$$u(0) = u'(0) = u''(0) = ... = u^{(n-1)}(0) = 0, u'(1) = 0,$$
$$u(0) = u'(0) = u''(0) = ... = u^{(n-2)}(0) = 0, u'(1) = 0,$$
$$u(0) = u'(0) = u''(0) = ... = u^{(n-2)}(0) = 0, u''(1) = 0$$

El-Shahed and Hassan [5] studied the existence of positive solutions of the q-difference boundary value problem:

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$$-D_q^2 u(t) = a(t) f(u(t)), \quad 0 \le t \le 1,$$
$$\alpha u(0) - \beta D_q u(0) = 0,$$
$$\gamma u(1) + \delta D_q u(1) = 0.$$

The purpose of this paper is to establish the existence of positive solutions to nonlinear nth order q `boundary value problems:

$$D_q^n u(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1, \tag{1}$$

$$u(0) = D_q^2 u(0) = D_q^3 u(0) = ... = D_q^{n-1} u(0) = 0, D_q u(1) = 0, \tag{2}$$

$$u(0) = D_q u(0) = D_q^2 u(0) = ... = D_q^{n-2} u(0) = 0, D_q u(1) = 0, \tag{3}$$

$$u(0) = D_q u(0) = D_q^2 u(0) = ... = D_q^{n-2} u(0) = 0, D_q^2 u(1) = 0, \tag{4}$$

where λ is a positive parameter. Throughout the paper, we assume that

C1: $f : [0, \infty) \to [0, \infty)$ is continuous

C2: $a : (0,1) \to [0, \infty)$ is continuous function such that $\int_0^1 a(t) d_q t > 0$.

2. PRELIMINARIES:

For the convenience of the reader, we present here some notations and lemmas that will be used in the proof our main results.

Let $q \in (0,1)$ and defined [14]

$$[a]_q = \frac{q^a - 1}{q - 1} = q^{a-1} + \ldots + 1, \quad a \in \Box .$$

The q-analogue of the power function $(a-b)^n$ with $n \in \Box$ is

$$(a-b)^0 = 1 \quad , \qquad (a-b)^n = \prod_{k=0}^{n-1} (a - bq^k), \quad a, b \in \Box , \quad n \in \Box .$$

More generally, if $\alpha \in \Box$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{i=0}^{\infty} \frac{(a - bq^i)}{(a - bq^{\alpha+i})}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \Box \setminus \{0, -1, -2, \ldots\}, \ 0 < q < 1,$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x},$$

and q-derivatives of higher order by

$$D_q^n f(x) = \begin{cases} f(x) & if \quad n = 0, \\ D_q D_q^{n-1} f(x) & if \quad n \in \mathbb{N}. \end{cases}$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$\int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad 0 \le |q| < 1, \quad x \in [0, b].$$

If $a \in [0, b]$ and f defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Similarly as done for derivatives, it can be defined an operator I_q^n, namely,

$$(I_q^0 f)(x) = f(x) \text{ and } (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [14]. We now point out three formulas that will be used later (${}_i D_q$ denotes the derivative with respect to variable i) [8]

$$[a(t-s)]^{(\alpha)} = a^{\alpha} (t-s)^{(\alpha)}, \tag{5}$$

$${}_t D_q (t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)}, \tag{6}$$

$$\left({}_x D_q \int_0^x f(x,t) d_q t \right)(x) = \int_0^x {}_x D_q f(x,t) d_q t + f(qx, x). \tag{7}$$

Remark: 2.1. We note that if $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$ [8].

Definition: 2.1. Let $\alpha \ge 0$ and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann–Liouville type is $({}_{RL} I_q^0 f)(x) = f(x)$ and

$$({}_{RL} I_q^{\alpha} f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^x (x - qt)^{(\alpha-1)} f(t) d_q t, \quad \alpha \in \mathbb{R}^+, x \in [0,1].$$

Definition: 2.2. [18] The fractional q-derivative of the Riemann–Liouville type of order $\alpha \ge 0$ is defined by $({}_{RL} D_q^0 f)(x) = f(x)$ and $({}_{RL} D_q^{\alpha} f)(x) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f)(x), \quad \alpha > 0$, where $[\alpha]$ is the smallest integer greater than or equal to α.

Definition: 2.3. [18] The fractional q-derivative of the Caputo type of order $\alpha \geq 0$ is defined by

$$({}_{C}D_{q}^{\alpha} f)(x) = (I_{q}^{[\alpha]-\alpha} D_{q}^{[\alpha]} f)(x), \quad \alpha > 0,$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

Lemma: 2.1. Let $\alpha, \beta \geq 0$ and f be a function defined on $[0,1]$. Then, the next formulas hold:

1. $(I_{q}^{\beta} I_{q}^{\alpha} f)(x) = (I_{q}^{\alpha+\beta} f)(x),$

2. $(D_{q}^{\alpha} I_{q}^{\alpha} f)(x) = f(x).$

The next result is important in the sequel. It was proved in a recent work by the author [8].

Theorem: 2.1. Let $\alpha > 0$ and p be a positive integer. Then, the following equality holds:

$$({}_{RL}I_{q}^{\alpha}\, {}_{RL}D_{q}^{p} f)(x) = (D_{q}^{p} I_{q}^{\alpha} f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_{q}(\alpha+k-p+1)} (D_{q}^{k} f)(0).$$

Theorem: 2.2. [18] Let $x > 0$ and $\alpha \in \mathbb{R}^{+} \setminus \mathbb{N}$. Then, the following equality holds:

$$(I_{q}^{\alpha}\, {}_{C}D_{q}^{\alpha} f)(x) = f(x) - \sum_{k=0}^{[\alpha]-1} \frac{x^{k}}{\Gamma_{q}(k+1)} (D_{q}^{k} f)(0).$$

Definition: 2.4. Let X be a real Banach space. A nonempty closed convex set $P \subset X$ is called cone of X if it satisfies the following conditions

1. $x \in P, \sigma \geq 0$ Implies $\sigma x \in P$;

.

2. $x \in P, -x \in P$ Implies $x = 0$

Definition: 2.5. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Theorem: 2.3. [10,15] Let X be a Banach space and $P \subset X$ is a cone in X. Assume that Ω_1 and Ω_2 are open subsets in X of with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be completely continuous operator. In addition suppose either:

H1: $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_2$ or
H2: $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_2$ and $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_1$,
holds. Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. GREEN FUNCTIONS AND THEIR PROPERTIES:

Lemma: 3.1. Let $y \in C[0,1]$, then the boundary value problem

$$D_{q}^{n} u_2(t) + y(t) = 0, \quad 0 < t < 1,$$

$$u_2(0) = D_{q}^{2} u_2(0) = D_{q}^{3} u_2(0) = \ldots\ldots = D_{q}^{n-1} u_2(0) = 0, D_{q} u_2(1) = 0,$$

has a unique solution

$$u_2(t)=\int_0^1 G_2(t,qs)\,y(s)\,d_qs,$$

where

$$G_2(t,s)=\begin{cases}\dfrac{t(1-s)^{n-2}}{\Gamma_q(n-1)}-\dfrac{(t-s)^{n-1}}{\Gamma_q(n)}, & 0\le s\le t\le 1,\\[2ex] \dfrac{t(1-s)^{n-2}}{\Gamma_q(n-1)}, & 0\le t\le s\le 1.\end{cases}$$

Proof: We may apply Lemma 2.1 and Theorem 2.2, we see that

$$u_2(t)=u_2(0)+\frac{D_qu_2(0)}{\Gamma_q(2)}t+\frac{D_q^2u_2(0)}{\Gamma_q(3)}t^2+\frac{D_q^3u_2(0)}{\Gamma_q(4)}t^3+......+\frac{D_q^{n-1}u_2(0)}{\Gamma_q(n)}t^{n-1}-I_q^{\alpha}y(t).$$

By using the boundary conditions $u_2(0)=D_q^2u_2(0)=D_q^3u_2(0)=......=D_q^{n-1}u_2(0)=0$, we get

$$u_2(t)=B_2t-\int_0^t\frac{(t-qs)^{n-1}}{\Gamma_q(n)}y(s)\,d_qs\quad. \tag{8}$$

Differentiating both sides of (8) one obtain, with the help (6) and (7),

$$(D_qu_2)(t)=B_2-\int_0^t\frac{[n-1]_q\,(t-qs)^{n-2}}{\Gamma_q(n)}y(s)\,d_qs\ ,$$

then by the condition $D_qu_2(1)=0,$ we have

$$B_2=\int_0^1\frac{(1-qs)^{n-2}}{\Gamma_q(n-1)}y(s)\,d_qs,$$

the proof is complete.

Lemma: 3.2. Function G_2 defined above satisfies the following conditions:

$$G_2(t,qs)\ge 0 \text{ and } G_2(t,qs)\le G_2(1,qs),\quad 0\le t,s\le 1, \tag{9}$$

$$G_2(t,qs)\ge \eta_2(t)G_2(1,qs),\quad 0\le t,s\le 1\quad with\quad \eta_2(t)=t. \tag{10}$$

Proof: We start by defining two functions

$$g_1(t,s)=\frac{t(1-s)^{n-2}}{\Gamma_q(n-1)}-\frac{(t-s)^{n-1}}{\Gamma_q(n)},\quad 0\le s\le t\le 1,$$

and

$$g_2(t,s)=\frac{t(1-s)^{n-2}}{\Gamma_q(n-1)},\quad 0\le t\le s\le 1.$$

It is clear that $g_2(t,qs)\ge 0$. Now, $g_1(0,qs)=0$ and, in view of Remark 2.1, for $t\ne 0$

$$g_1(t,qs)=\frac{t\,(1-qs)^{n-2}}{\Gamma_q(n-1)}-\frac{(t-qs)^{n-1}}{\Gamma_q(n)}$$
$$\geq\frac{t\,(1-qs)^{n-2}}{\Gamma_q(n-1)}-\frac{t\,(1-qs)^{n-1}}{\Gamma_q(n)}$$
$$=\frac{t}{\Gamma_q(n)}\Big[\,[n-1]_q\,(1-q\,s)^{n-2}-(1-q\,s)^{n-1}\Big]\geq 0,$$

therefore, $G_2(t,qs)\geq 0$. Moreover, for fixed $s\in[0,1]$,

$${}_tD_q g_1(t,qs)=\frac{(1-q\,s)^{n-2}}{\Gamma_q(n-1)}-\frac{[n-1]_q\,(t-q\,s)^{n-2}}{\Gamma_q(n)}$$
$$=\frac{[n-1]_q\left[(1-q\,s)^{n-2}-(t-q\,s)^{n-2}\right]}{\Gamma_q(n)}\geq 0,$$

i.e., $g_1(t,qs)$ is an increasing function of t. Obviously, $g_2(t,qs)$ is increasing in t, therefore $G_2(t,qs)$ is an increasing function of t for fixed $s\in[0,1]$. This concludes the proof of (9).

Suppose now that $t\geq qs$, Then

$$\frac{G_2(t,qs)}{G_2(1,qs)}=\frac{[n-1]_q\,t\,(1-qs)^{n-2}-(t-qs)^{n-1}}{[n-1]_q\,(1-qs)^{n-2}-(1-qs)^{n-1}}$$
$$\geq\frac{[n-1]_q\,t\,(1-q\,s)^{n-2}-t\,(1-q\,s)^{n-1}}{[n-1]_q\,(1-q\,s)^{n-2}-(1-q\,s)^{n-1}}=t.$$

If $t\leq qs$. Then

$$\frac{G_2(t,qs)}{G_2(1,qs)}=\frac{t\,(1-q\,s)^{n-2}/\Gamma_q(n-1)}{G_2(1,qs)}$$
$$>\frac{t\,(1-q\,s)^{n-2}/\Gamma_q(n-1)}{(1-q\,s)^{n-2}/\Gamma_q(n-1)}=t,$$

and this finishes the proof of (10).

Lemma: 3.3. Let $y\in C[0,1]$, then the q-boundary value problem

$$D_q^n u_3(t)+y(t)=0,\quad 0<t<1,$$

$$u_3(0)=D_q u_3(0)=D_q^2 u_3(0)=......=D_q^{n-2}u_3(0)=0,D_q u_3(1)=0,$$

has a unique solution

$$u_3(t)=\int_0^1 G_3(t,qs)\,y(s)\,d_qs,$$

where

$$G_3(t,s)=\begin{cases}\dfrac{t^{n-1}(1-s)^{n-2}}{\Gamma_q(n)}-\dfrac{(t-s)^{n-1}}{\Gamma_q(n)}, & 0\le s\le t\le 1,\\[2mm] \dfrac{t^{n-1}(1-s)^{n-2}}{\Gamma_q(n)}, & 0\le t\le s\le 1.\end{cases}$$

Proof: We may apply Lemma 2.1 and Theorem 2.2, we see that

$$u_3(t)=u_3(0)+\frac{D_qu_3(0)}{\Gamma_q(2)}t+\frac{D_q^2u_3(0)}{\Gamma_q(3)}t^2+......+\frac{D_q^{n-1}u_3(0)}{\Gamma_q(n)}t^{n-1}-I_q^{\alpha}y(t).$$

By using the boundary conditions $u_3(0)=D_qu_3(0)=D_q^2u_3(0)=......=D_q^{(n-2)}u_3(0)=0$, we get

$$u_3(t)=B_3t^{n-1}-\int_0^t\frac{(t-qs)^{n-1}}{\Gamma_q(n)}y(s)d_qs\ . \tag{11}$$

Differentiating both sides of (11) one obtain, with the help (6) and (7),

$$(D_qu_3)(t)=B_3\left[n-1\right]_q t^{n-2}-\int_0^t\frac{\left[n-1\right]_q(t-qs)^{n-2}}{\Gamma_q(n)}y(s)d_qs\ ,$$

then by the condition $D_qu_3(1)=0,$ we have

$$B_3=\int_0^1\frac{(1-qs)^{n-2}}{\Gamma_q(n)}y(s)d_qs,$$

the proof is complete.

Lemma: 3.4. Function G_3 defined above satisfies the following conditions:

$$G_3(t,qs)\ge 0 \text{ and } G_3(t,qs)\le G_3(1,qs),\quad 0\le t,s\le 1, \tag{12}$$

$$G_3(t,qs)\ge \eta_3(t)G_3(1,qs),\quad 0\le t,s\le 1\quad with\quad \eta_3(t)=t^{n-1}. \tag{13}$$

Proof: We start by defining two functions

$$g_3(t,s)=\frac{t^{n-1}(1-s)^{n-2}}{\Gamma_q(n)}-\frac{(t-s)^{n-1}}{\Gamma_q(n)},\quad 0\le s\le t\le 1,$$

and

$$g_4(t,s)=\frac{t^{n-1}(1-s)^{n-2}}{\Gamma_q(n)},\quad 0\le t\le s\le 1.$$

It is clear that $g_4(t,qs)\ge 0$. Now, $g_3(0,qs)=0$ and, in view of Remark 2.1, for $t\neq 0$

$$\begin{aligned}g_3(t,qs)&=\frac{t^{n-1}(1-qs)^{n-2}}{\Gamma_q(n)}-\frac{(t-qs)^{n-1}}{\Gamma_q(n)}\\ &\ge\frac{t^{n-1}(1-qs)^{n-2}}{\Gamma_q(n)}-\frac{t^{n-1}(1-qs)^{n-1}}{\Gamma_q(n)}\\ &=\frac{t^{n-1}}{\Gamma_q(n)}\left[(1-q\,s)^{n-2}-(1-q\,s)^{n-1}\right]\ge 0,\end{aligned}$$

therefore, $G_3(t,qs) \geq 0$. Moreover, for fixed $s \in [0,1]$,

$$
{}_tD_q g_3(t,qs) = \frac{[n-1]_q t^{n-2}(1-qs)^{n-2} - [n-1]_q (t-qs)^{n-2}}{\Gamma_q(n)}
$$

$$
\geq \frac{[n-1]_q t^{n-2}(1-qs)^{n-2} - [n-1]_q t^{n-2}(1-qs)^{n-2}}{\Gamma_q(n)} = 0,
$$

i.e., $g_3(t,qs)$ is an increasing function of t. Obviously, $g_4(t,qs)$ is increasing in t, therefore $G_3(t,qs)$ is an increasing function of t for fixed $s \in [0,1]$. This concludes the proof of (12).

Suppose now that $t \geq qs$, Then

$$
\frac{G_3(t,qs)}{G_3(1,qs)} = \frac{t^{n-1}(1-qs)^{n-2} - (t-qs)^{n-1}}{(1-qs)^{n-2} - (1-qs)^{n-1}}
$$

$$
\geq \frac{t^{n-1}(1-qs)^{n-2} - t^{n-1}(1-qs)^{n-1}}{(1-qs)^{n-2} - (1-qs)^{n-1}} = t^{n-1}.
$$

If $t \leq qs$. Then

$$
\frac{G_3(t,qs)}{G_3(1,qs)} = \frac{t^{n-1}(1-qs)^{n-2} \big/ \Gamma_q(n)}{G_3(1,qs)}
$$

$$
> \frac{t^{n-1}(1-qs)^{n-2} \big/ \Gamma_q(n)}{(1-qs)^{n-2} \big/ \Gamma_q(n)} = t^{n-1},
$$

and this finishes the proof of (13).

Lemma: 3.5. Let $y \in C[0,1]$, then the q-boundary value problem

$$
D_q^n u_4(t) + y(t) = 0, \quad 0 < t < 1,
$$

$$
u_4(0) = D_q u_4(0) = D_q^2 u_4(0) = \ldots\ldots = D_q^{n-2} u_4(0) = 0, D_q^2 u_4(1) = 0,
$$

has a unique solution

$$
u_4(t) = \int_0^1 G_4(t,qs)\, y(s)\, d_q s,
$$

where

$$
G_4(t,s) = \begin{cases} \dfrac{t^{n-1}(1-s)^{n-3}}{\Gamma_q(n)} - \dfrac{(t-s)^{n-1}}{\Gamma_q(n)}, & 0 \leq s \leq t \leq 1, \\[2ex] \dfrac{t^{n-1}(1-s)^{n-3}}{\Gamma_q(n)}, & 0 \leq t \leq s \leq 1. \end{cases}
$$

The proof of Lemma 3.5 is very similar to that of Lemma 3.3 and therefore omitted.

Lemma: 3.6. Function G_4 defined above satisfies the following conditions:

$$G_4(t,qs) \ge 0 \text{ and } G_4(t,qs) \le G_4(1,qs), \quad 0 \le t, s \le 1, \tag{14}$$

$$G_4(t,qs) \ge \eta_4(t) G_4(1,qs), \quad 0 \le t, s \le 1 \quad with \quad \eta_4(t) = t^{n-1}. \tag{15}$$

Proof: We start by defining two functions

$$g_5(t,s) = \frac{t^{n-1}(1-s)^{n-3}}{\Gamma_q(n)} - \frac{(t-s)^{n-1}}{\Gamma_q(n)}, \quad 0 \le s \le t \le 1,$$

and

$$g_6(t,s) = \frac{t^{n-1}(1-s)^{n-3}}{\Gamma_q(n)}, \quad 0 \le t \le s \le 1.$$

It is clear that $g_6(t,qs) \ge 0$. Now, $g_5(0,qs) = 0$ and, in view of Remark 2.1, for $t \ne 0$

$$\begin{aligned} g_5(t,qs) &= \frac{t^{n-1}(1-qs)^{n-3}}{\Gamma_q(n)} - \frac{(t-qs)^{n-1}}{\Gamma_q(n)} \\ &\ge \frac{t^{n-1}(1-qs)^{n-3}}{\Gamma_q(n)} - \frac{t^{n-1}(1-qs)^{n-1}}{\Gamma_q(n)} \\ &= \frac{t^{n-1}}{\Gamma_q(n)}\left[(1-qs)^{n-3} - (1-qs)^{n-1} \right] \ge 0, \end{aligned}$$

therefore, $G_4(t,qs) \ge 0$. Moreover, for fixed $s \in [0,1]$,

$$\begin{aligned} {}_tD_q g_5(t,qs) &= \frac{[n-1]_q t^{n-2}(1-qs)^{n-3} - [n-1]_q (t-qs)^{n-2}}{\Gamma_q(n)} \\ &\ge \frac{t^{n-2}(1-qs)^{n-3} - t^{n-2}(1-qs)^{n-2}}{\Gamma_q(n-1)} \\ &= \frac{t^{n-2}}{\Gamma_q(n-1)}\left[(1-qs)^{n-3} - (1-qs)^{n-2} \right] \ge 0, \end{aligned}$$

i.e., $g_5(t,qs)$ is an increasing function of t. Obviously, $g_6(t,qs)$ is increasing in t, therefore $G_4(t,qs)$ is an increasing function of t for fixed $s \in [0,1]$. This concludes the proof of (14).

Suppose now that $t \ge qs$, Then

$$\begin{aligned} \frac{G_4(t,qs)}{G_4(1,qs)} &= \frac{t^{n-1}(1-qs)^{n-3} - (t-qs)^{n-1}}{(1-qs)^{n-3} - (1-qs)^{n-1}} \\ &\ge \frac{t^{n-1}(1-qs)^{n-3} - t^{n-1}(1-qs)^{n-1}}{(1-qs)^{n-3} - (1-qs)^{n-1}} = t^{n-1}. \end{aligned}$$

If $t \le qs$. Then

$$\frac{G_4(t,qs)}{G_4(1,qs)} = \frac{t^{n-1}(1-qs)^{n-3}/\Gamma_q(n)}{G_4(1,qs)}$$
$$> \frac{t^{n-1}(1-qs)^{n-3}/\Gamma_q(n)}{(1-qs)^{n-3}/\Gamma_q(n)} = t^{n-1},$$

and this finishes the proof of (15).

4. Main results:

In this section, we will apply Krasnoselskii's fixed point theorem to the eigenvalue problem (1), (i) (i=2,3,4).

Remark: 3.1 : If we let $0 < \tau < 1$, then

$$\min_{t\in[\tau,1]} G_i(t,qs) \geq \eta_i(\tau) G_i(1,qs), \quad for \quad s \in [0,1]. \tag{16}$$

Let $X = C[0,1]$ be the Banach space endowed with norm $\|u_i\| = \max_{t\in[\tau,1]} |u_i(t)|$. Let $\tau = q^n$ [9] for a given $n \in □$ and define the cone $P \subset X$ by

$$P = \left\{ u_i \in X : u_i(t) \geq 0, \min_{t\in[\tau,1]} u_i(t) \geq \eta_i(\tau)\|u_i\| \right\}.$$

Remark: 3.2: It follows from the non-negativeness and continuity of G_i, a and f that the operator $T : P \to X$ defined by

$$Tu_i(t) = \lambda \int_0^1 G_i(t,qs) a(s) f(u_i(s)) d_q s,$$

is completely continuous. Moreover, for $u_i \in P$, $(Tu_i)(t) \geq 0$ on $[0,1]$ and

$$\min_{t\in[\tau,1]} (Tu_i)(t) = \min_{t\in[\tau,1]} \lambda \int_0^1 G_i(t,qs)\, a(s) f(u_i(s)) d_q s$$
$$\geq \eta_i(\tau) \int_0^1 G_i(1,qs)\, a(s) f(u_i(s)) d_q s$$
$$= \eta_i(\tau) \|Tu_i\|,$$

that is, $T(P) \subset P$.

For our purposes, let us define two constants

$$\gamma = \left(\lambda \int_0^1 G_i(1,qs) a(s) d_q s \right)^{-1} \quad and \quad \beta = \left(\eta_i(\tau) \lambda \int_\tau^1 G_i(1,qs) a(s) d_q s \right)^{-1}.$$

Our existence result is now presented.

Theorem: 3.1. Let $\tau = q^n$ with $n \in □$. Suppose that $f(u_i)$ is a nonnegative continuous function on $[0,1] \times [0,\infty)$. If there exist two positive constants $R > r > 0$ such that

$$\max_{(s,u_i)\in[0,1]\times[0,r]} f(u_i(t)) \leq \gamma u_i, \tag{17}$$

$$\min_{(s,u_i)\in[\tau,1]\times[\eta_i(\tau)R,R]} f(u_i(t)) \geq \beta u_i, \tag{18}$$

then problem (1)–(4) has a solution u_i satisfying $u_i(t)>0$ for $t\in(0,1]$.

Proof: Since the operator $T:P\to X$ is completely continuous we only have to show that the operator equation $u_i=Tu_i$ has a solution satisfying $u_i(t)>0$ for $t\in(0,1]$.

Let $\Omega_1=\{u_i\in X:\|u_i\|<r\}$. For $u_i\in P\cap\partial\Omega_1$, we have $0\le u_i(t)\le r$ on $[0,1]$. Using (9),(12),(14) and (17) we obtain,

$$\begin{aligned}\|Tu_i\|&=\max_{t\in[0,1]}\lambda\int_0^1 G_i(t,qs)a(s)f(u_i(s))d_qs\\&\le\lambda\int_0^1 G_i(1,qs)a(s)\max_{(s,u_i)\in[0,1]\times[0,r]}f(u_i(s))d_qs\\&\le\gamma r\lambda\int_0^1 G_i(1,qs)a(s)d_qs\\&=r=\|u_i\|.\end{aligned}$$

Let $\Omega_2=\{u_i\in X:\|u_i\|<R\}$. For $u_i\in P\cap\partial\Omega_2$, we have $\eta_i(\tau)R_2\le u(t)\le R$ on $[\tau,1]$. Using (16) and (18), and the fact that $\tau=q^n$ [9], we obtain

$$\begin{aligned}\|Tu_i\|&=\max_{t\in[0,1]}\lambda\int_0^1 G_i(t,qs)a(s)f(u_i(s))d_qs\\&\ge\lambda\int_\tau^1 G_i(1,qs)a(s)\min_{(s,u_i)\in[0,\tau]\times[\eta_i(\tau)R,R]}f(u_i(s))d_qs\\&\ge\beta\eta_i(\tau)R\lambda\int_\tau^1 G_i(1,qs)a(s)d_qs\\&=R=\|u_i\|.\end{aligned}$$

Now, Theorem 3.1 assures the existence of a fixed point u_i of T such that $r\le\|u_i\|\le R$. To finish the proof, note that by (10),(13) and (15)

$$\begin{aligned}u_i(t)&=\lambda\int_0^1 G_i(t,qs)a(s)f(u_i(s))d_qs\\&\ge\eta_i(t)\lambda\int_0^1 G_i(1,qs)a(s)f(u_i(s))d_qs\\&=\eta_i(t)\|u_i\|,\end{aligned}$$

which implies that $u_i(t)\ge\eta_i(t)r$. Therefore, $u_i(t)>0$ for $t\in(0,1]$ and the proof is done.

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